

A generalization of heterochromatic graphs

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Abstract

In 2006, Suzuki, and Akbari & Alipour independently presented a necessary and sufficient condition for edge-colored graphs to have a heterochromatic spanning tree, where a heterochromatic spanning tree is a spanning tree whose edges have distinct colors. In this paper, we propose *f-chromatic* graphs as a generalization of heterochromatic graphs. An edge-colored graph is *f-chromatic* if each color c appears on at most $f(c)$ edges. We also present a necessary and sufficient condition for edge-colored graphs to have an *f-chromatic* spanning forest with exactly m components. Moreover, using this criterion, we show that a g -chromatic graph G of order n with $|E(G)| > \binom{n-m}{2}$ has an *f-chromatic* spanning forest with exactly m ($1 \leq m \leq n-1$) components if $g(c) \leq \frac{|E(G)|}{n-m} f(c)$ for any color c .

Keyword(s): *f*-chromatic, heterochromatic, rainbow, multicolored, totally multicolored, polychromatic, colorful, edge-coloring, k -bounded coloring, spanning tree, spanning forest.

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1 Introduction

We consider finite undirected graphs without loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. An *edge-coloring* of a graph G is a mapping $color : E(G) \rightarrow \mathbb{C}$, where \mathbb{C} is a set of colors. An *edge-colored graph* $(G, \mathbb{C}, color)$ is a graph G with an edge-coloring $color$ on a color set \mathbb{C} . We often abbreviate an edge-colored graph $(G, \mathbb{C}, color)$ as G .

An edge-colored graph G is said to be *heterochromatic* if no two edges of G have the same color, that is, $color(e) \neq color(f)$ for any two distinct edges e and f of G . A heterochromatic graph is also said to be *rainbow*, *multicolored*, *totally multicolored*, *polychromatic*, or *colorful*. Heterochromatic

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¹05C05 Trees.

²05C15 Coloring of graphs and hypergraphs.

subgraphs of edge-colored graphs have been studied in many papers. (See the survey by Kano and Li [7].)

We begin with some results for the existence of heterochromatic spanning trees and forests. Brualdi and Hollingsworth [3] showed the following theorem and conjecture for edge-disjoint heterochromatic spanning trees in complete graphs.

Theorem 1.1 (Brualdi and Hollingsworth, (1996) [3]). *If the complete graph K_{2n} ($n \geq 3$) is edge-colored in such a way that each color induces a perfect matching, then it has two edge-disjoint heterochromatic spanning trees.*

Conjecture 1.2 ([3]). *Under the same condition as in Theorem 1.1, the edges of K_{2n} can be partitioned into n edge-disjoint heterochromatic spanning trees.*

Suzuki [8] presented a necessary and sufficient condition for general connected graphs to have a heterochromatic spanning tree. Here, we denote by $\omega(G)$ the number of components of a graph G . Given an edge-colored graph G and a color set R , we define $E_R(G) = \{e \in E(G) \mid \text{color}(e) \in R\}$. Similarly, for a color c , we define $E_c(G) = E_{\{c\}}(G)$. We denote the graph $(V(G), E(G) \setminus E_R(G))$ by $G - E_R(G)$.

Theorem 1.3 (Suzuki, (2006) [8]). *An edge-colored connected graph G has a heterochromatic spanning tree if and only if*

$$\omega(G - E_R(G)) \leq |R| + 1 \quad \text{for any } R \subseteq \mathbb{C}.$$

Jin and Li [6] generalized this theorem to the following theorem, from which we can obtain Theorem 1.3 by taking $k = n - 1$.

Theorem 1.4 (Jin and Li, (2006) [6]). *An edge-colored connected graph G of order n has a spanning tree with at least k ($1 \leq k \leq n - 1$) colors if and only if*

$$\omega(G - E_R(G)) \leq n - k + |R| \quad \text{for any } R \subseteq \mathbb{C}.$$

If an edge-colored connected graph G of order n has a spanning tree with at least k colors, then G has a heterochromatic spanning forest with k edges, that is, G has a heterochromatic spanning forest with exactly $n - k$ components. On the other hand, If an edge-colored connected graph G of order n has a heterochromatic spanning forest with exactly $n - k$ components, then the forest can be turned into a spanning tree with at least k colors by adding some $n - k - 1$ edges. Hence, we can rephrase Theorem 1.4 as the following.

Theorem 1.5 ([6]). *An edge-colored connected graph G of order n has a heterochromatic spanning forest with exactly $n - k$ components ($1 \leq k \leq n - 1$) if and only if*

$$\omega(G - E_R(G)) \leq n - k + |R| \quad \text{for any } R \subseteq \mathbb{C}.$$

Akbari and Alipour [1] gave another necessary and sufficient condition for graphs to have a heterochromatic spanning tree.

Theorem 1.6 (Akbari and Alipour, (2006) [1]). *An edge-colored connected graph G of order n has a heterochromatic spanning tree if and only if for every partition of $V(G)$ into t ($1 \leq t \leq n$) parts, there exist at least $t - 1$ edges with distinct colors that join different partition sets.*

Theorem 1.3 and Theorem 1.6 are essentially the same, but the proofs are different. Theorem 1.3 was proved graph theoretically, and Theorem 1.6 was proved by using Rado's Theorem in Matroid Theory.

Suzuki [8] proved the following theorem by applying Theorem 1.3.

Theorem 1.7 (Suzuki, (2006) [8]). *An edge-colored complete graph K_n has a heterochromatic spanning tree if $|E_c(G)| \leq n/2$ for any color $c \in \mathbb{C}$.*

This theorem implies that by properly bounding numbers of edges for each color, the graph can contain enough colors to have a heterochromatic spanning tree. If the edges of a graph G is colored so that no color appears on more than k edges, we refer to this as a k -bounded edge-coloring. Erdős, Nešetřil and Rödl [4] mentioned the following problem.

Problem 1.8 (Erdős, Nešetřil and Rödl, (1983) [4]). *Find a bound $k = k(n)$ such that every k -bounded edge-colored complete graph K_n contains a heterochromatic Hamiltonian cycle.*

Hahn and Thomassen [5] proved the following theorem.

Theorem 1.9 (Hahn and Thomassen, (1986) [5]). *There exists a constant number c such that if $n \geq ck^3$ then every k -bounded edge-colored complete graph K_n has a heterochromatic Hamiltonian cycle.*

Albert, Frieze, and Reed [2] improved Theorem 1.9.

Theorem 1.10 (Albert, Frieze, and Reed, (1995) [2]). *Let $c < 1/32$. If n is sufficiently large and $k \leq \lceil cn \rceil$, then every k -bounded edge-colored complete graph K_n has a heterochromatic Hamiltonian cycle.*

In this paper, we will propose a generalization of heterochromatic graphs, which is also a generalization of k -bounded colored graphs. Moreover, we will generalize Theorem 1.3, 1.5, 1.7, and Problem 1.8.

2 Heterochromatic and f -chromatic graphs

Heterochromatic or *k -bounded colored* means that any color appears at most *once* or *k* times, respectively. We propose to generalize *once* and *k* to a mapping f from a given color set \mathbb{C} to the set of non-negative integers. We introduce the following definition as a generalization of heterochromatic or k -bounded colored graphs³.

definition 2.1. Let f be a mapping from a given color set \mathbb{C} to the set of non-negative integers. An edge-colored graph $(G, \mathbb{C}, \text{color})$ is said to be f -chromatic if $|E_c(G)| \leq f(c)$ for any color $c \in \mathbb{C}$.

Fig.1 shows an example of an f -chromatic spanning tree of an edge-colored graph. Let $\mathbb{C} = \{1, 2, 3, 4, 5, 6, 7\}$ be a given color set of 7 colors, and a mapping f is given as follows: $f(1) = 3$, $f(2) = 1$, $f(3) = 3$, $f(4) = 0$, $f(5) = 0$, $f(6) = 1$, $f(7) = 2$. Then, the left edge-colored graph in Fig.1 has the right graph as a subgraph. It is a spanning tree where each color c appears at most $f(c)$ times. Thus, it is an f -chromatic spanning tree.

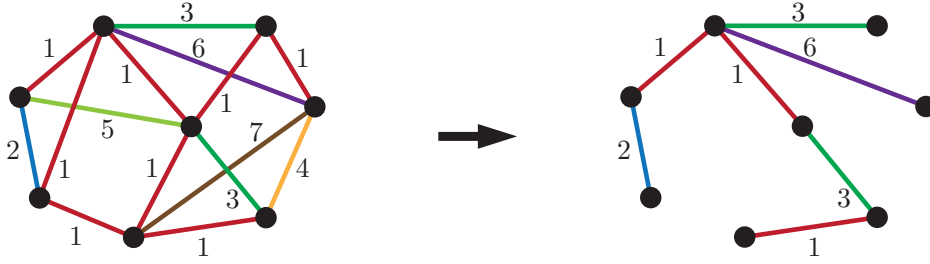


Fig. 1: An f -chromatic spanning tree of an edge-colored graph.

If $f(c) = 1$ for any color c , then all f -chromatic graphs are heterochromatic and also all heterochromatic graphs are f -chromatic. We expect many previous studies and results for heterochromatic subgraphs will be generalized. For example, we give the following generalization of Problem 1.8.

Problem 2.2. Find a relationship between two functions f and g such that every g -chromatic complete graph K_n contains an f -chromatic Hamiltonian cycle (Hamiltonian path, spanning tree, or other subgraph).

In this paper, we generalize Theorems 1.3, 1.5, and 1.7. Let \mathbb{C} be a color set, and f be a mapping from \mathbb{C} to the set of non-negative integers. We present the following necessary and sufficient condition for graphs to have an f -chromatic spanning forest with exactly m components.

³We name it after *heterochromatic*. Of course, we may name it as *f -bounded colored*.

Theorem 2.3. *An edge-colored graph $(G, \mathbb{C}, \text{color})$ of order at least m has an f -chromatic spanning forest with exactly m components if and only if*

$$\omega(G - E_R(G)) \leq m + \sum_{c \in R} f(c) \quad \text{for any } R \subseteq \mathbb{C}.$$

Note that, it is allowed $R = \emptyset$, that is, the condition includes the necessary condition that $\omega(G) \leq m$ to have a spanning forest with m components. From this theorem, we can obtain the following corollary.

Corollary 2.4. *Let $\mathbb{C} = \{c_1, c_2, \dots, c_r\}$ and f be a mapping from \mathbb{C} to the set of non-negative integers such that $\sum_{i=1}^r f(c_i) = n - m$. An edge-colored graph $(G, \mathbb{C}, \text{color})$ of order n has an spanning forest with exactly m components and exactly $f(c_i)$ edges for each color c_i if and only if*

$$\omega(G - E_R(G)) \leq m + \sum_{c \in R} f(c) \quad \text{for any } R \subseteq \mathbb{C}.$$

This corollary is interesting because in the corollary we can not only desire an f -chromatic spanning forest, but also fix the exactly number of edges for each color.

Theorem 2.3 will be proved in the next section. By applying Theorem 2.3, we will prove the following theorem.

Theorem 2.5. *A g -chromatic graph G of order n with $|E(G)| > \binom{n-m}{2}$ has an f -chromatic spanning forest with exactly m ($1 \leq m \leq n-1$) components if $g(c) \leq \frac{|E(G)|}{n-m} f(c)$ for any color c .*

In order to prove Theorem 2.5, we need the following Lemma.

Lemma 2.6. *Let G be a graph of order n that consists of s components. Then $|E(G)| \leq \binom{n-(s-1)}{2}$.*

Proof. Take a graph G^* with the maximum number of edges that satisfies the condition in the Lemma. By the maximality of G^* , each component is complete. Let D_s be a maximum component of G^* . Suppose that some component D except D_s has at least two vertices. Let x be a vertex of D . Let $D' = D - x$ and $D'_s = (V(D_s) \cup \{x\}, E(D_s) \cup \{xz \mid z \in D_s\})$. Then, we have $|E(D)| + |E(D_s)| < |E(D')| + |E(D'_s)|$, which contradicts the maximality of $E(G^*)$. Thus, every component except D_s has exactly one vertex, which implies that $|V(D_s)| = n - (s - 1)$. Therefore, $|E(G)| \leq |E(G^*)| = \binom{n-(s-1)}{2}$. ■

We here prove Theorem 2.5 using Lemma 2.6.

Proof. Suppose that G has no f -chromatic spanning forests with exactly m components. By Theorem 2.3, there exists a color set $R \subseteq \mathbb{C}$ such that

$$\omega(G - E_R(G)) > m + \sum_{c \in R} f(c).$$

Let $s = \omega(G - E_R(G))$ and $r = \sum_{c \in R} f(c)$. Then we have

$$n \geq s > m + r. \quad (1)$$

Let D_1, D_2, \dots, D_s be the components of $G - E_R(G)$, and q be the number of edges of G between these distinct components. Note that, the colors of these q edges are only in R . By the assumption on the function g , we get

$$q \leq \sum_{c \in R} g(c) \leq \sum_{c \in R} \frac{|E(G)|}{n-m} f(c) = \frac{|E(G)|}{n-m} r. \quad (2)$$

On the other hand,

$$q = |E(G)| - |E(D_1) \cup E(D_2) \cup \dots \cup E(D_s)|.$$

By Lemma 2.6, $|E(D_1) \cup E(D_2) \cup \dots \cup E(D_s)| \leq \binom{n-(s-1)}{2}$. Thus, since $s \geq r + m + 1$ by (1), we have

$$\begin{aligned} q &= |E(G)| - |E(D_1) \cup E(D_2) \cup \dots \cup E(D_s)| \\ &\geq |E(G)| - \binom{n-(s-1)}{2} \geq |E(G)| - \binom{n-(r+m)}{2} \\ &= |E(G)| - \frac{(n-r-m)(n-r-m-1)}{2}. \end{aligned}$$

Hence, by (2),

$$|E(G)| - \frac{(n-r-m)(n-r-m-1)}{2} \leq \frac{|E(G)|}{n-m} r,$$

and thus,

$$\frac{n-m-r}{n-m} |E(G)| \leq \frac{(n-r-m)(n-r-m-1)}{2}.$$

Since $n-r-m > 0$ by (1), by dividing both sides of the equation by $n-r-m$

$$\frac{|E(G)|}{n-m} \leq \frac{n-r-m-1}{2},$$

and thus,

$$|E(G)| \leq \frac{(n-m)(n-r-m-1)}{2}.$$

Since $r \geq 0$, we get $|E(G)| \leq \binom{n-m}{2}$, which contradicts our assumption. Therefore, G has an f -chromatic spanning forest with m components. \blacksquare

We can obtain the following corollary from Theorem 2.5.

Corollary 2.7. *A g -chromatic complete graph K_n has an f -chromatic spanning forest with exactly m ($1 \leq m \leq n-1$) components if $g(c) \leq \frac{n(n-1)}{2(n-m)}f(c)$ for any color c .*

Theorems 1.3 and 1.7 are special cases of Theorem 2.3 and Corollary 2.7 with $m = 1$ and $f(c) = 1$. Theorem 1.5 is a special case of Theorem 2.3 for connected graphs with $m = n - k$ and $f(c) = 1$. The proof of Theorem 2.5 is essentially the same as the proof of Theorem 1.7 in [8]. We can also prove Theorem 2.3 in a similar way as the proof of Theorem 1.3 in [8]. However, in this paper, we will improve the proof by introducing the new notion of *Saturated Conditions*.

3 Proof of Theorem 2.3

We begin with some notation. We use the symbol \subset to denote proper inclusion. We often denote an edge $e = \{x, y\}$ by xy or yx . For a graph G and a subset $E \subseteq E(G)$, we denote the graphs $(V(G), E(G) \setminus E)$ and $(V(G), E(G) \cup E)$ by $G - E$ and $G + E$, respectively. Similarly, for an edge e whose end vertices are in $V(G)$, we denote the graphs $(V(G), E(G) \setminus \{e\})$ and $(V(G), E(G) \cup \{e\})$ by $G - e$ and $G + e$, respectively. For an edge-colored graph $(G, \mathbb{C}, \text{color})$ and an edge set $E \subseteq E(G)$, we define $\text{color}(E) = \{\text{color}(e) \mid e \in E\}$ and $\text{color}(G) = \text{color}(E(G))$.

First, we prove the necessity. Let F be an f -chromatic spanning forest of G with exactly m components. Consider $G - E_R(G)$ for any color subset $R \subseteq \mathbb{C}$. Since $F - E_R(F)$ is a spanning forest of $G - E_R(G)$, we have $\omega(G - E_R(G)) \leq \omega(F - E_R(F))$. Moreover, $\omega(F - E_R(F)) = m + |E_R(F)|$ because F is a forest with exactly m components. By the definition of f -chromatic graphs, $|E_c(F)| \leq f(c)$ for any color c . Thus, we have

$$|E_R(F)| = \sum_{c \in R} |E_c(F)| \leq \sum_{c \in R} f(c).$$

Hence,

$$\omega(G - E_R(G)) \leq \omega(F - E_R(F)) = m + |E_R(F)| \leq m + \sum_{c \in R} f(c).$$

Next, we prove the sufficiency by contradiction. Suppose that G has no f -chromatic spanning forests with exactly m components.

Claim 1. *Any f -chromatic spanning forest of G has at least $m + 1$ components.*

Proof. If there exists an f -chromatic spanning forest of G with at most $m - 1$ components, then it can be turned into an f -chromatic spanning forest of G with exactly m components by removing edges one by one, which contradicts our assumption. ■

We denote by E_F^* the set of edges between the components of a forest F in G , namely,

$$E_F^* := \{xy \in E(G) \mid x \text{ and } y \text{ are not in the same components of } F\}.$$

Fig.2 shows an example of E_F^* . Let D_1, D_2, \dots, D_7 be the components of $G - E_F^*$, which is induced by the components of F . We simplify the left graph to the right illustration.

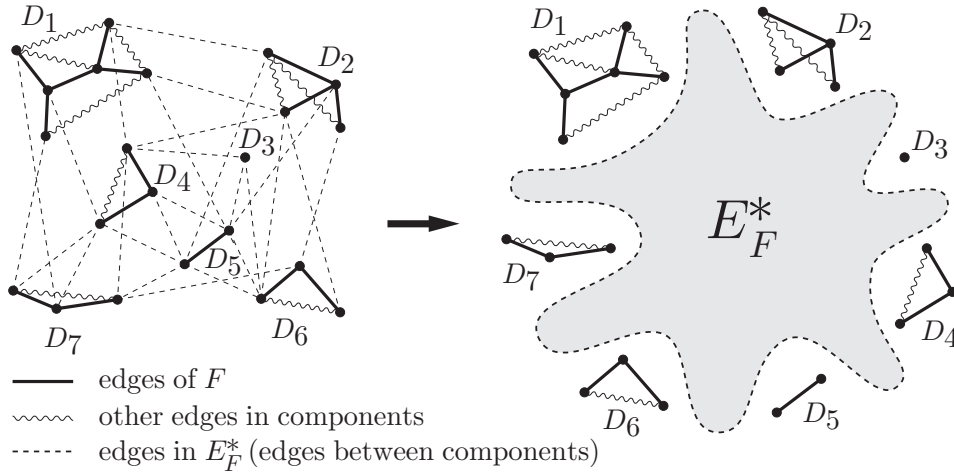


Fig. 2: An example of E_F^* , which is the set of edges between the components of a forest F in G .

For an f -chromatic spanning forest F of G and two sets of colors C_0, C_1 , the triple $\langle F, C_0, C_1 \rangle$ is said to be *saturated* if the following conditions hold:

Saturated Conditions

- (i) $C_0 \cap C_1 = \emptyset$,
- (ii) $C_0 \cup C_1 = \text{color}(E_F^*)$,
- (iii) $C_0 \cap \text{color}(F) = \emptyset$,
- (iv) $\omega(F) \geq m + 1 + \sum_{c \in C_0} f(c)$,
- (v) $|E_c(\tilde{F})| = f(c)$ for every color $c \in C_1$, where \tilde{F} is any f -chromatic spanning forest of G such that $E_{\tilde{F}}^* = E_F^*$ and $C_0 \cap \text{color}(\tilde{F}) = \emptyset$.

Saturated Conditions imply that if already \tilde{F} has $f(c)$ edges for every color $c \in C_1$, then we can not add more edges in E_F^* whose color appears on \tilde{F} in order to get a larger f -chromatic spanning forest, namely, we call that *saturated*. Note that, it is allowed that $\tilde{F} = F$ in the condition (v).

Claim 2. *There exists a saturated triple $\langle F, C_0, C_1 \rangle$ in G .*

Proof. G has an f -chromatic spanning forest, because the graph $(V(G), \emptyset)$ is an f -chromatic spanning forest. Let F be an f -chromatic spanning forest with minimum $\omega(F)$, and let

$$\begin{aligned} C_0 &:= \text{color}(E_F^*) \setminus \text{color}(F), \\ C_1 &:= \text{color}(E_F^*) \setminus C_0. \end{aligned}$$

Then, the triple $\langle F, C_0, C_1 \rangle$ satisfies the saturated conditions (i), (ii) and (iii). Suppose that there exists a color $c \in C_0$ such that $f(c) \geq 1$. By the definition of C_0 , there exists some edge $e \in E_F^*$ with the color c . By the saturated condition (iii), the spanning forest $F + e$ is f -chromatic, which contradicts the minimality of $\omega(F)$. Thus, $f(c) = 0$ for any color $c \in C_0$. Then, by Claim 1, we have

$$\omega(F) \geq m + 1 = m + 1 + 0 = m + 1 + \sum_{c \in C_0} f(c).$$

Hence, the triple $\langle F, C_0, C_1 \rangle$ satisfies the saturated condition (iv).

Let \tilde{F} be any f -chromatic spanning forest of G such that $E_{\tilde{F}}^* = E_F^*$ and $C_0 \cap \text{color}(\tilde{F}) = \emptyset$. Note that \tilde{F} may be F . By the definition of f -chromatic graphs, $|E_c(\tilde{F})| \leq f(c)$ for any color $c \in C_1 \subseteq \mathbb{C}$. Suppose that there exists a color $c \in C_1$ such that $|E_c(\tilde{F})| < f(c)$. By the definition of C_1 , there exists some edge $e \in E_F^*$ with the color c . Since $E_{\tilde{F}}^* = E_F^*$, the edge e is also in $E_{\tilde{F}}^*$. Then, the spanning forest $\tilde{F} + e$ is f -chromatic, which contradicts the minimality of $\omega(F)$. Thus, $|E_c(\tilde{F})| = f(c)$ for any color $c \in C_1$. Hence, the triple $\langle F, C_0, C_1 \rangle$ satisfies the saturated condition (v). Therefore, $\langle F, C_0, C_1 \rangle$ is a saturated triple in G . ■

Let $\langle F, C_0, C_1 \rangle$ be a saturated triple with maximal C_0 in G .

Claim 3. $C_1 \neq \emptyset$.

Proof. Suppose $C_1 = \emptyset$. By the saturated condition (ii) of $\langle F, C_0, C_1 \rangle$, $C_0 = C_0 \cup C_1 = \text{color}(E_F^*)$. Then, $E_F^* \subseteq E_{C_0}(G)$. Hence, by the saturated condition (iv) of $\langle F, C_0, C_1 \rangle$, we have

$$\omega(G - E_{C_0}(G)) \geq \omega(G - E_F^*) = \omega(F) \geq m + 1 + \sum_{c \in C_0} f(c),$$

which contradicts the assumption of the theorem. ■

By the definition of a saturated triple, F is an f -chromatic spanning forest of G . We define a triple $\langle F', C'_0, C'_1 \rangle$ based on $\langle F, C_0, C_1 \rangle$ as follows:

$$\begin{aligned} F' &:= F - E_{C_1}(F), \\ C'_0 &:= C_0 \cup C_1, \\ C'_1 &:= \text{color}(E_{F'}^*) \setminus C'_0. \end{aligned}$$

Fig.3 shows an example of E_F^* and $E_{F'}^*$ where the left illustration is the same as in Fig.2. By removing edges in $E_{C_1}(F)$ from F , some components of F splits into several new components. $E_{F'}^*$ is the set of edges between these new components and edges in E_F^* , that is, the set of edges between components of F' in G .

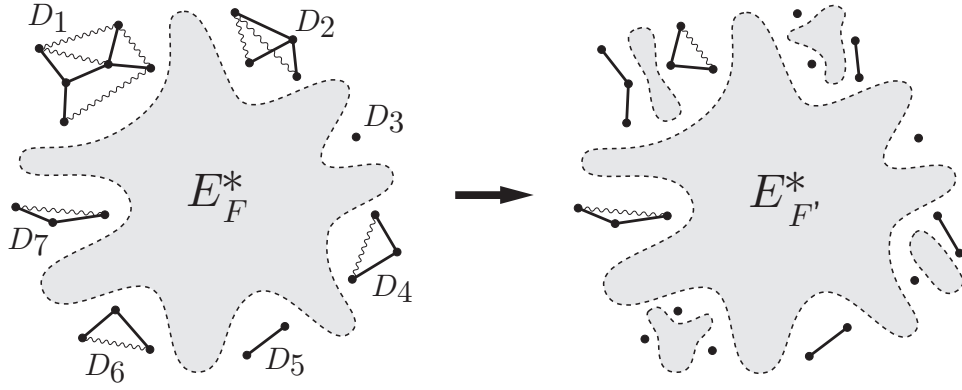


Fig. 3: An example of E_F^* and $E_{F'}^*$.

Since $C_1 \neq \emptyset$ by Claim 3, we have $C_0 \subset C_0 \cup C_1 = C'_0$, that is, C'_0 properly contains C_0 . Thus, if we can prove that $\langle F', C'_0, C'_1 \rangle$ is saturated then it contradicts the maximality of C_0 and Theorem 2.3 is proved. Note that F' is an f -chromatic spanning forest of G because F is an f -chromatic spanning forest of G .

Claim 4. $\langle F', C'_0, C'_1 \rangle$ satisfies the saturated conditions (i), (ii), and (iii).

Proof. The triple $\langle F', C'_0, C'_1 \rangle$ satisfies the saturated condition (i), that is, $C'_0 \cap C'_1 = \emptyset$ by the definition of C'_0 and C'_1 .

By the definition of F' , we have $\text{color}(E_F^*) \subseteq \text{color}(E_{F'}^*)$. By the definition of C'_0 and the condition (ii) of the saturated triple $\langle F, C_0, C_1 \rangle$, we have $C'_0 = C_0 \cup C_1 = \text{color}(E_F^*) \subseteq \text{color}(E_{F'}^*)$. Thus, by the definition of C'_1 , the triple $\langle F', C'_0, C'_1 \rangle$ satisfies the saturated condition (ii).

By the condition (iii) of the saturated triple $\langle F, C_0, C_1 \rangle$, we have $C_0 \cap \text{color}(F) = \emptyset$, so $C_0 \cap \text{color}(F') = \emptyset$ because $\text{color}(F') \subseteq \text{color}(F)$. By

the definition of F' , we have $C_1 \cap \text{color}(F') = \emptyset$. Hence, by the definition of C'_0 , the triple $\langle F', C'_0, C'_1 \rangle$ satisfies the saturated condition (iii). ■

Claim 5. $\langle F', C'_0, C'_1 \rangle$ satisfies the saturated condition (iv).

Proof. By the definition of F' , we have

$$\omega(F') = \omega(F) + \sum_{c \in C_1} |E_c(F)|. \quad (3)$$

By the saturated conditions (iv) and (v) of $\langle F, C_0, C_1 \rangle$, we have

$$\omega(F) + \sum_{c \in C_1} |E_c(F)| \geq m + 1 + \sum_{c \in C_0} f(c) + \sum_{c \in C_1} f(c). \quad (4)$$

By the saturated condition (i) of $\langle F, C_0, C_1 \rangle$ and the definition of C'_0 , we have

$$\sum_{c \in C_0} f(c) + \sum_{c \in C_1} f(c) = \sum_{c \in C_0 \cup C_1} f(c) = \sum_{c \in C'_0} f(c). \quad (5)$$

Thus, by the equations and inequalities (3), (4), and (5), we have

$$\omega(F') \geq m + 1 + \sum_{c \in C'_0} f(c).$$

Hence, the triple $\langle F', C'_0, C'_1 \rangle$ satisfies the saturated condition (iv). ■

In order to prove the last saturated condition (v), we need some preparation. Let \tilde{F}' be any f -chromatic spanning forest of G such that $E_{\tilde{F}'}^* = E_{F'}^*$ and $C'_0 \cap \text{color}(\tilde{F}') = \emptyset$. By the definition of F' , $F = F' + E_{C_1}(F)$. We want to consider the graph $H = \tilde{F}' + E_{C_1}(F)$ instead of F .

Fig.4 shows how to construct the graph H from F . First, we get F' by removing the edges in $E_{C_1}(F)$ from F . Next, by changing edges only inside components of $G - E_{F'}^*$, we pick up any f -chromatic spanning forest \tilde{F}' of G such that $E_{\tilde{F}'}^* = E_{F'}^*$ and $C'_0 \cap \text{color}(\tilde{F}') = \emptyset$. Last, we get H by adding back the edges in $E_{C_1}(F)$ to \tilde{F}' , which are indicated by double lines.

Claim 6. H is an f -chromatic spanning forest of G such that $E_H^* = E_F^*$ and $C_0 \cap \text{color}(H) = \emptyset$.

Proof. H is a spanning subgraph of G because \tilde{F}' is a spanning forest of G . Let T_1, T_2, \dots, T_k and $\tilde{T}_1, \tilde{T}_2, \dots, \tilde{T}_l$ be the components of F' and \tilde{F}' , respectively, which are trees. The graphs $G - E_{F'}^*$ and $G - E_{\tilde{F}'}^*$ consist of k and l components induced by T_i 's and \tilde{T}_i 's, respectively. Since $E_{\tilde{F}'}^* = E_{F'}^*$, $G - E_{F'}^* = G - E_{\tilde{F}'}^*$. Thus, $k = l$ and we may assume that $V(T_i) = V(\tilde{T}_i)$ for every i . Hence, an edge $e \in E_{C_1}(F)$ connects T_i and T_j in G , if and only if e

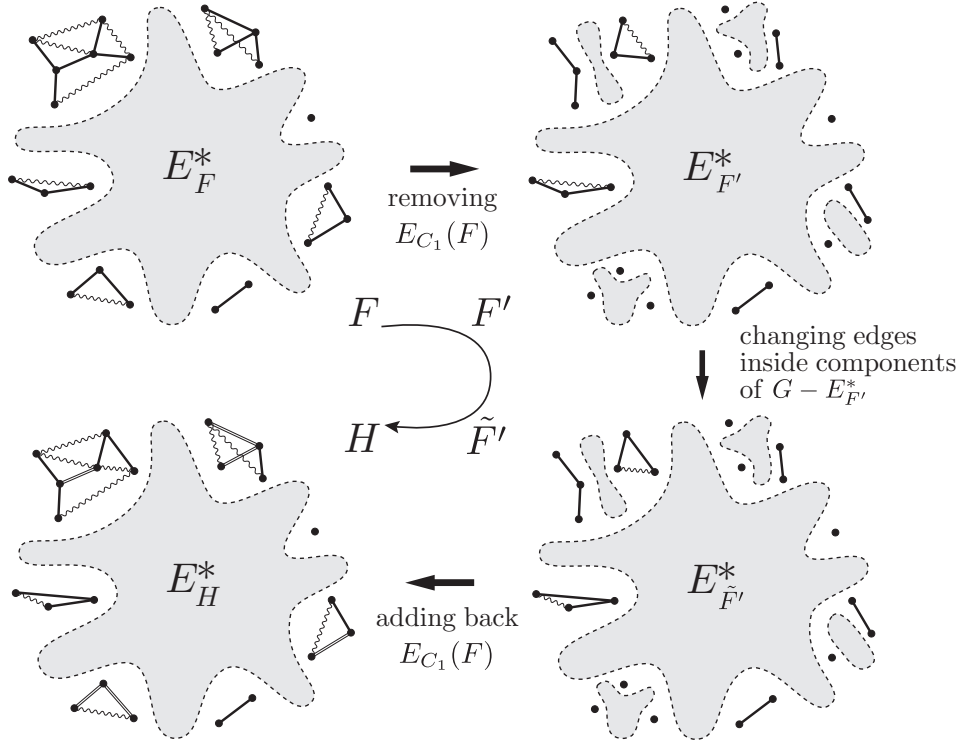


Fig. 4: How to construct H from F .

connects \tilde{T}_i and \tilde{T}_j in G . Therefore, since F is a forest, H also is a spanning forest of G and $E_H^* = E_F^*$.

Since $C'_0 \cap \text{color}(\tilde{F}') = \emptyset$ and $C'_0 = C_0 \cup C_1$, \tilde{F}' has no colors of $E_{C_1}(F)$. Thus, $H = \tilde{F}' + E_{C_1}(F)$ is f -chromatic because both \tilde{F}' and F are f -chromatic. Since $C'_0 \cap \text{color}(\tilde{F}') = \emptyset$ and $C'_0 = C_0 \cup C_1$, \tilde{F}' has no colors in C_0 . By the saturated condition (i) of $\langle F, C_0, C_1 \rangle$, $E_{C_1}(F)$ has no colors in C_0 . Hence, $C_0 \cap \text{color}(H) = C_0 \cap \text{color}(\tilde{F}' + E_{C_1}(F)) = \emptyset$. ■

Claim 7. $\langle F', C'_0, C'_1 \rangle$ satisfies the saturated condition (v).

Proof. Suppose that the triple $\langle F', C'_0, C'_1 \rangle$ does not satisfy the saturated condition (v), namely, there exists some color $c \in C'_1$ such that $|E_c(\tilde{F}')| \neq f(c)$ for some f -chromatic spanning forest \tilde{F}' of G such that $E_{\tilde{F}'}^* = E_{F'}^*$ and $C'_0 \cap \text{color}(\tilde{F}') = \emptyset$. Then, $|E_c(\tilde{F}')| < f(c)$ because \tilde{F}' is f -chromatic. By the definition of C'_1 , $E_{F'}^*$ has some edge e with the color c .

First, we show that $e \notin E(H)$ and $e \notin E_H^*$. Since $E_{\tilde{F}'}^* = E_{F'}^*$, $e \in E_{\tilde{F}'}^*$, which implies $e \notin E(\tilde{F}')$. By the definition of C'_1 and C'_0 , and the saturated condition (ii) of $\langle F, C_0, C_1 \rangle$, $c \notin C'_0 = C_0 \cup C_1 = \text{color}(E_F^*)$, so $e \notin E_{C_1}(F)$ and $e \notin E_F^*$. Then, we have shown the following subclaims.

Subclaim 7.1. $c \notin C'_0 = C_0 \cup C_1 = \text{color}(E_F^*)$.

Subclaim 7.2. $e \in E_{\tilde{F}'}^*$, $e \notin E(\tilde{F}')$, $e \notin E_{C_1}(F)$, and $e \notin E_F^*$.

Thus, $e \notin E(\tilde{F}') \cup E_{C_1}(F) = E(H)$, and $e \notin E_H^*$ because $E_F^* = E_H^*$ by Claim 6.

Subclaim 7.3. $e \notin E(H)$ and $e \notin E_H^*$.

Hence, by Subclaim 7.3, the edge e connects two vertices x and y in the same tree component T of H , and $H + e$ has a cycle D . Since $e \in E_{\tilde{F}'}^*$ by Subclaim 7.2, e connects two tree components \tilde{T}_i and \tilde{T}_j of \tilde{F}' for some i and j . Note that both \tilde{T}_i and \tilde{T}_j are subgraphs of T because $e \notin E_H^*$ by Subclaim 7.3. Thus, there exists a path connecting x and y without e in H , and the cycle D consists of such a path and e . Hence, the cycle D contains some edge $e' \in E(H) \setminus E(\tilde{F}') = E_{C_1}(F)$ by the definition of H . Note that $T + e - e'$ is a tree with $V(T) = V(T + e - e')$. Let $\tilde{F} = H + e - e'$ and $c' = \text{color}(e')$. Then, $c \neq c'$ because $c' \in C_1$ and $c \notin C_1$ by Subclaim 7.1. By Claim 6, H is an f -chromatic spanning forest of G . Since $H = \tilde{F}' + E_{C_1}(F)$ and $c \notin C_1$ by Subclaim 7.1, $|E_c(H)| = |E_c(\tilde{F}')| < f(c)$ by the assumption $|E_c(\tilde{F}')| < f(c)$. Thus, $\tilde{F} = H + e - e'$ is also an f -chromatic spanning forest of G . Since $V(T) = V(T + e - e')$, $E_{\tilde{F}}^* = E_H^*$, that is, $E_{\tilde{F}}^* = E_F^*$ by Claim 6. Moreover, $c \notin C_0$ by Subclaim 7.1. Thus, By Claim 6, $C_0 \cap \text{color}(\tilde{F}) = C_0 \cap \text{color}(H + e - e') = \emptyset$. Hence, \tilde{F} is an f -chromatic spanning forest of G such that $E_{\tilde{F}}^* = E_F^*$ and $C_0 \cap \text{color}(\tilde{F}) = \emptyset$. On the other hand, since $c \neq c' \in C_1$, $|E_{c'}(\tilde{F})| = |E_{c'}(H)| - 1 < |E_{c'}(H)| \leq f(c')$, which contradicts the saturated condition (v) of $\langle F, C_0, C_1 \rangle$. ■

By Claim 4, 5, and 7, $\langle F', C'_0, C'_1 \rangle$ is saturated. As discussed above, it contradicts the maximality of C_0 . Consequently, Theorem 2.3 is proved.

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